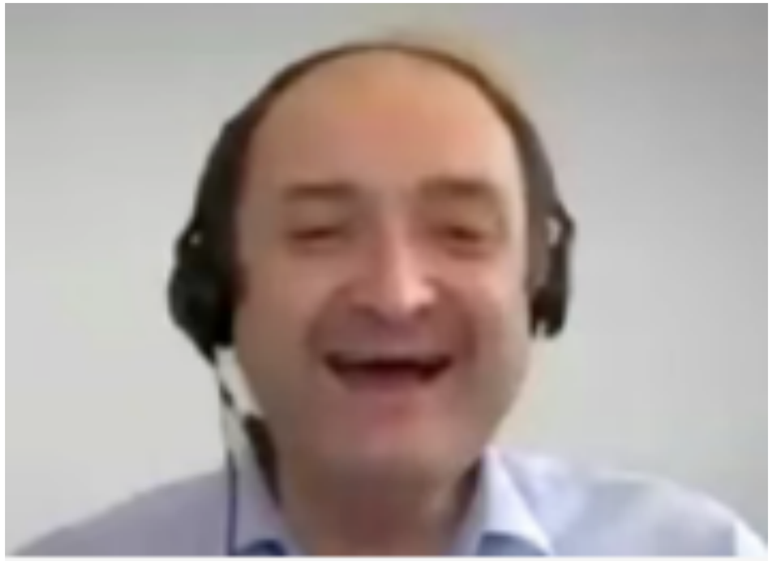
ECE4010J Final RC

Shao Yujie



Non-parametric Single Sample Test for Median

- Sign Test for the Median
- Flexible

20.2. Sign Test. Let $X_1, \ldots, X_n$ be a random sample of size n from an arbitrary continuous distribution and let

$$Q_+ = \#\{X_k : X_k - M_0 > 0\}, \qquad Q_- = \#\{X_k : X_k - M_0 < 0\}.$$

We reject at significance level α

- $H_0 : M \le M_0$ if $P[Q_- \le k \mid M = M_0] < \alpha$,
- $H_0 : M \ge M_0$ if $P[Q_+ \le k \mid M = M_0] < \alpha$,
- $H_0 : M = M_0$ if $P[\min(Q_-, Q_+) \le k \mid M = M_0] < \alpha/2$.

Non-parametric Single Sample Test for Median

- Wilcoxon Signed Rank Test

20.4. Wilcoxon Signed Rank Test. Let $X_1, \ldots, X_n$ be a random sample of size n from a symmetric distribution. Order the n absolute differences $|X_i - M|$ according to magnitude, so that $X_{R_i} - M_0$ is the R_ith smallest difference by modulus. If ties in the rank occur, the mean of the ranks is assigned to all equal values.

Let

$$W_+ = \sum_{R_i>0} R_i, \qquad |W_-| = \sum_{R_i<0} |R_i|.$$

We reject at significance level α

- $H_0: M \le M_0$ if $|W_-|$ is smaller than the critical value for α,
- $H_0: M \ge M_0$ if W_+ is smaller than the critical value for α,
- $H_0: M = M_0$ if $W = \min(W_+, |W_-|)$ is smaller than the critical value for $\alpha/2$.

20.5. Example. Returning to the previous example, we want to test $H_0: M \le 3.5$ and have the following observations, ordered from smallest to largest:

X_i	$X_i - M_0$	R_i	X_i	$X_i - M_0$	R_i
3	-0.5	-5.5	2	-1.5	-13
3	-0.5	-5.5	5	1.5	$+13$
3	-0.5	-5.5	5	1.5	$+13$
3	-0.5	-5.5	5	1.5	$+13$
4	0.5	$+5.5$	5	1.5	$+13$
4	0.5	$+5.5$	1	-2.5	-18
4	0.5	$+5.5$	6	2.5	$+18$
4	0.5	$+5.5$	6	2.5	$+18$
4	0.5	$+5.5$	6	2.5	$+18$
4	0.5	$+5.5$	6	2.5	$+18$

We calculate the sum of the negative ranks,

$$w_- = -5.5 - 5.5 - 5.5 - 5.5 - 13 - 18 = -53.$$

Consulting a table, the critical value for $n = 20$ and $\alpha = 0.05$ is 60. For $\alpha = 0.01$ it is 43. Since $|w_-|$ lies between these values, the P-value of the test is between 1% and than 5%, most likely around 2%-3%.

n	p 0.05	0.025	0.01	0.005
18	47	40	33	28
19	54	46	38	32
20	60	52	43	37

Non-parametric Single Sample Test for Median

- Wilcoxon Signed Rank Test Example

Exercise 1.

A company wants to test whether a new assembly line procedure increases the physical stress on its workers. It selects eleven workers to work for one day using each of the assembly line procedures. At the end of each day, their pulse frequency is measured:

Procedure 1	X	63	65	71	75	72	75	68	74	62	73	72
Procedure 2	Y	80	78	96	87	88	96	82	83	77	79	71

It is thought that the median pulse frequency is higher in Procedure 2 than in Procedure 1.

We perform a paired test and consider M_{Y-X}. Then we test

$$H_0: M_{Y-X} \le 0.$$ ← M0

We will reject H_0 if $|W_-|$ is small. **(1/2 Mark)** We calculate $Y - X$:

$Y-X$	17	13	25	12	16	21	14	9	15	6	-1

(1/2 Mark) We can see from the table of $Y - X$ that there is only a single negative value of $Y - X$, which has rank 1. Therefore,

$$|W_-| = 1.$$

What is W+?

W+ = 2+3+4+…+10

(1/2 Mark) and $W = |W_-| = 1$. **(1/2 Mark)** According to the table for the Wilcoxon signed-rank test, we reject H_0 at the 5% level of significance if $W < 14$, so we can here reject H_0. **(1/2 Mark)**

n	p 0.05	0.025	0.01	0.005
5	1			
6	2	1		
7	4	2	0	
8	6	4	2	0
9	8	6	3	2
10	11	8	5	3
11	14	11	7	5
12	17	14	10	7
13	21	17	13	10
14	26	21	16	13
15	30	25	20	16
16	36	30	24	19
17	41	35	28	23

Test for Proportion

- When to use: we want to test the proportion of members of a population with some trait (or sample mean for n i.i.d. Bernoulli variables) p_0.
- What distribution does the testing statistics follow: Given a sample $X_1, \ldots, X_n$, $\frac{\overline{X}-p_0}{\sqrt{p_0(1-p_0)/n}}$ can be approximated by standard normal distribution Z (by Central Limit Theorem) for large sample size n.
- We reject at significance level α
  $H_0 : p = p_0$ if $|Z| > z_{\alpha/2}$,
  $H_0 : p \leq p_0$ if $Z > z_{\alpha}$,
  $H_0 : p \geq p_0$ if $Z < -z_{\alpha}$.

Test for Comparing Two Proportions

- When to use: we want compare two sample means for two groups (number:n_1, n_2) of Bernoulli variables (p_1, p_2) and test their differences $(p_1 - p_2)_0$
- What distribution does the testing statistics follow: Given two samples $X_{1_1}, \dots, X_{1_n}$,$X_{2_1}, \dots, X_{2_n}$, $\frac{\overline{X_1} - \overline{X_2} - (p_1 - p_2)}{\sqrt{\overline{X_1}(1-\overline{X_1})/n_1 + \overline{X_2}(1-\overline{X_2})/n_2}}$ can be approximated by standard normal distribution Z (by Central Limit Theorem) for large sample size n_1, n_2.
- We reject at significance level α
  $H_0 : p_1 - p_2 = (p_1 - p_2)_0$ if $|Z| > z_{\alpha/2}$,
  $H_0 : p_1 - p_2 \leq (p_1 - p_2)_0$ if $Z > z_{\alpha}$,
  $H_0 : p_1 - p_2 \geq (p_1 - p_2)_0$ if $Z < -z_{\alpha}$.
- Pooled test for Comparing Two Proportions: define **Pooled estimator** $\hat{p}$:$(n_1\hat{p}_1 + n_2\hat{p}_2)/(n_1 + n_2)$, the test statistics become: $\frac{\hat{P}}{\sqrt{\hat{p}(1-\hat{p})(1/n_1 + 1/n_2)}}$ (In pooled test, we assume $p_1 - p_2 = 0$)

F-Distribution

Definition

Let $\chi^2_{\gamma_1}$ and $\chi^2_{\gamma_2}$ be independent chi-squared random variables with γ_1 and γ_2 degrees of freedom, respectively. Then the random variable

$$F_{\gamma_1,\gamma_2} = \frac{\chi^2_{\gamma_1}/\gamma_1}{\chi^2_{\gamma_2}/\gamma_2}$$

follows a **F-distribution** with γ_1 and γ_2 degrees of freedom.

Property

$$P[F_{\gamma_1,\gamma_2} < x] = P\left[\frac{1}{F_{\gamma_1,\gamma_2}} > \frac{1}{x}\right] = 1 - P\left[F_{\gamma_2,\gamma_1} < \frac{1}{x}\right]$$

Comparing Variances

Given two **normally-distributed** populations $X^{(1)} \sim N(\mu_1, \sigma_1^2)$ and $X^{(2)} \sim N(\mu_2, \sigma_2^2)$, we conduct an **F-test** to compare σ_1^2 and σ_2^2.

Procedure

Note that **normality** is required in comparison of variances.

1. Set up H_0 (and H_1 if applicable).
2. Set up **F-statistics** $F_{n_1-1,n_2-1} = \frac{S_1^2}{S_2^2}$.
3. Decide whether to reject H_0: we reject at significance level α
   - $H_0 : \sigma_1 \leq \sigma_2$ if $S_1^2/S_2^2 > f_{\alpha,n_1-1,n_2-1}$
   - $H_0 : \sigma_1 \geq \sigma_2$ if $S_2^2/S_1^2 > f_{\alpha,n_2-1,n_1-1}$
   - $H_0 : \sigma_1 = \sigma_2$ if $S_1^2/S_2^2 > f_{\alpha/2,n_1-1,n_2-1}$ or $S_2^2/S_1^2 > f_{\alpha/2,n_2-1,n_1-1}$

OC Curve

For $n_1 = n_2 = n$, the abscissa is defined by $\lambda = \frac{\sigma_1}{\sigma_2}$.

Three Basic Cases

For two **normally distributed** populations:

- $X^{(1)} \sim N(\mu_1, \sigma_1^2)$
- $X^{(2)} \sim N(\mu_2, \sigma_2^2)$

When comparing μ_1 and μ_2, we consider three basic cases:

- σ_1^2 and σ_2^2 are known
- σ_1^2 and σ_2^2 are unknown but $\sigma_1^2 = \sigma_2^2$
- σ_1^2 and σ_2^2 are unknown and not necessarily equal

Case 1: Variance Known

Procedure

1. Set up H_0 (and H_1 if applicable).
2. Set up **Z-statistics** $Z = \frac{\overline{X}^{(1)} - \overline{X}^{(2)} - (\mu_1 - \mu_2)_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$.
3. Decide whether to reject H_0: we reject at significance level α
   - $H_0 : \mu_1 - \mu_2 = (\overline{\mu_1 - \mu_2})_0$ if $|Z| > z_{\alpha/2}$
   - $H_0 : \mu_1 - \mu_2 \leq (\mu_1 - \mu_2)_0$ if $Z > z_\alpha$
   - $H_0 : \mu_1 - \mu_2 \geq (\mu_1 - \mu_2)_0$ if $Z < -z_\alpha$

OC Curve

- We can use the OC curves for normal distributions with $n = n_1 = n_2$ with $d = \frac{|\mu_1 - \mu_2|}{\sqrt{\sigma_1^2 + \sigma_2^2}}$.
- When $n_1 \neq n_2$, we use the equivalent sample size $n = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2/n_1 + \sigma_2^2/n_2}$.

Case 2: Variance Equal but Unknown

Suppose we have random samples of sizes n_1, n_2 of $X^{(1)}$ and $X^{(2)}$, respectively and $X^{(1)} \sim N(\mu_1, \sigma^2), X^{(2)} \sim N(\mu_2, \sigma^2)$ with σ unknown.

- **Pooled estimator for variance**

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

- **T-statistics**

$$T_{n_1+n_2-2} = \frac{\overline{X}^{(1)} - \overline{X}^{(2)} - (\mu_1 - \mu_2)}{\sqrt{S_p^2(1/n_1 + 1/n_2)}}$$

- $100(1-\alpha)\%$ two-sided **confidence interval** for $\mu_1 - \mu_2$

$$\overline{X}^{(1)} - \overline{X}^{(2)} \pm t_{\alpha/2, n_1+n_2-2}\sqrt{S_p^2(1/n_1 + 1/n_2)}$$

Case 2: Variance Equal but Unknown

Procedure

1. Set up H_0 (and H_1 if applicable).
2. Calculate the **pooled estimator for variance** $S_p^2 = \frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$
3. Set up **T-statistics** $T_{n_1+n_2-2} = \frac{\overline{X}^{(1)}-\overline{X}^{(2)}-(\mu_1-\mu_2)_0}{\sqrt{S_p^2(1/n_1+1/n_2)}}$.
4. Decide whether to reject H_0: we reject at significance level α
   - $H_0 : \mu_1 - \mu_2 = (\mu_1 - \mu_2)_0$ if $|T_{n_1+n_2-2}| > t_{\alpha/2,n_1+n_2-2}$
   - $H_0 : \mu_1 - \mu_2 \leq (\mu_1 - \mu_2)_0$ if $T_{n_1+n_2-2} > t_{\alpha,n_1+n_2-2}$
   - $H_0 : \mu_1 - \mu_2 \geq (\mu_1 - \mu_2)_0$ if $T_{n_1+n_2-2} < -t_{\alpha,n_1+n_2-2}$

Case 2: Variance Equal but Unknown

OC Curve

- We can use the OC curves for the T-test with $n = n_1 = n_2$ with $d = \frac{|\mu_1 - \mu_2|}{2\sigma}$
- We must use the modified sample size $n^* = 2n - 1$ when reading the charts.

*Note: When σ is unknown, we must either

- use an estimate (like S_p) or
- express the deviation in terms of σ

Case 3: Variance Not Necessarily Equal and Unknown

The distribution of $S_1^2/n_1 + S_2^2/n_2$ is unknown, so we need to use the **Welch-Satterthwaite approximation** to approximate the distribution and perform **Welch's T-Test**.

Procedure

1. Set up H_0 (and H_1 if applicable).
2. Calculate the **approximate degrees of freedom** $\gamma = \frac{(S_1^2/n_1+S_2^2/n_2)^2}{\frac{(S_1^2/n_1)^2}{n_1-1}+\frac{(S_2^2/n_2)^2}{n_2-1}}$
3. Set up **T-statistics** $T_\gamma = \frac{\overline{X}^{(1)}-\overline{X}^{(2)}-(\mu_1-\mu_2)_0}{\sqrt{S_1^2/n_1+S_2^2/n_2}}$.
4. Decide whether to reject H_0: we reject at significance level α
   - $H_0 : \mu_1 - \mu_2 = (\mu_1 - \mu_2)_0$ if $|T_\gamma| > t_{\alpha/2,\gamma}$
   - $H_0 : \mu_1 - \mu_2 \leq (\mu_1 - \mu_2)_0$ if $T_\gamma > t_{\alpha,\gamma}$
   - $H_0 : \mu_1 - \mu_2 \geq (\mu_1 - \mu_2)_0$ if $T_\gamma < -t_{\alpha,\gamma}$

Case 3: Variance Not Necessarily Equal and Unknown

Remarks

- The choice of γ. We round γ down to the nearest integer.
- The choice of test when σ_1^2 and σ_2^2 are unknown. We always use Welch's test. Welch's test is only slightly less powerful than Student's test even if the variances are equal; if they are unequal,Student's test is very unreliable.
- No simple OC curves for Welch's test.

Thank You !!!

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